7. ELLIPSES

§7.1. The Equation of an Ellipse

An **ellipse** is the locus (set) of all points P such that the distance of P from a fixed point F, called a **focus**, and the perpendicular distance of P from a fixed line, called a **directrix**, satisfies the equation PF = ePM, where M is the foot of the perpendicular from P to the directrix and where e is a fixed real number with $0 < e < 1$.

Let a focus be $(f, 0)$ and let the directrix be $x = d$. If P(x, y) lies on the ellipse:

$$(x - f)^2 + y^2 = e^2(d - x)^2.$$

Hence $x^2 - 2xf + f^2 + y^2 = e^2d^2 - 2e^2xd + e^2x^2$

$\therefore\ x^2(1 - e^2) + y^2 - 2x(f - e^2d) = e^2d^2 - f^2$.

If we position the y-axis so that $f = e^2d$ then the equation simplifies to $x^2(1 - e^2) + y^2 = e^2d^2(1 - e^2)$.

It cuts the x-axis when $x = \pm\, ed$.

Let $a = ed$ and $b^2 = a^2(1 - e^2)$. Then $f = ae$ and $d = \frac{a}{e}$.

Then the equation becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This is symmetric about both the x- and y- axes and so there will be another focus and another directrix on the left of the y-axis.

The foci are at $(\pm\, ae, 0)$ and the directrices are $x = \pm\, \frac{a}{e}$.

The ellipse cuts the axes at $(\pm\, a, 0)$ and $(0, \pm\, b)$.

The line segment from $(-\, a, 0)$ to $(a, 0)$ is called the **major axis** and the line segment form $(0, -\, b)$ to (a, b) is called the **minor axis**.

Theorem 1: Let F_1 and F_2 be the foci of an ellipse and let P lie on the ellipse. Then $PF_1 + PF_2$ is the length of the major axis.

Proof: Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let e be the eccentricity. The foci are $F_1(ae, 0)$ and $F_2(-\, ae, 0)$.

The directrices are $x = \frac{a}{e}$ and $x = -\, \frac{a}{e}$.

Let P lie on the ellipse and let M_1 and M_2 be the foot of the perpendiculars to $x = \frac{a}{e}$ and $x = -\, \frac{a}{e}$ respectively.

$$PF_1 + PF_2 = e(PM_1 + PM_2) = e\frac{2a}{e} = 2a.$$

§7.2. Tangents to an Ellipse

The simplest way to obtain the equation of the tangents to an ellipse is to take it as the limit of a chord. This is sort of pre-calculus. If we were explaining this to our disembodied angel we wouldn't have to embark on a full-blown explanation of calculus. But we would have had to explain limits in discussing power series.

Theorem 2: The equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at $P(x_0, y_0)$ is $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$.

Proof: Let $Q(x_1, y_1)$ be another point on the conic, where $x_1 \neq x_0$. We might be tempted to say that x_1 is *close to* x_0, because we are going to take the limit as $x_1 \rightarrow x_0$, though this is not really necessary. Although we want it to be close enough to x_0 so that certain denominators are non-zero.

Since P and Q both lie on the conic,

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \text{ and}$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Subtracting we get: $\dfrac{x_0^2 - x_1^2}{a^2} + \dfrac{y_0^2 - y_1^2}{b^2} = 0.$

Hence $(x_0 - x_1)\left[\dfrac{x_0 + x_1}{a^2}\right] + (y_0 - y_1)\left[\dfrac{y_0 + y_1}{b^2}\right] = 0.$

$$\therefore \frac{y_0 - y_1}{x_0 - x_1} = -\frac{(x_0 + x_1)b^2}{(y_0 + y_1)a^2}.$$

Hence the equation of the chord PQ is

$$y - y_0 = = -\left(\frac{(x_0 + x_1)b^2}{(y_0 + y_1)a^2}\right)(x - x_0).$$

That is, $(y - y_0)(y_0 + y_1)\mathrm{a}^2 + (x - x_0)(x_0 + x_1)b^2 = 0$.

This can be simplified to $a^2yy_0 + b^2xx_0 = y_0{}^2a^2 + x_0{}^2b^2$

$= a^2b^2$ since P lies on the ellipse. Dividing by a^2b^2 we get the result.

Theorem 3: If $y = mx + c$ is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then $c = \pm\, a\sqrt{m^2 + b^2}$.

Proof: The equation of the ellipse can be written as

$$b^2x^2 + a^2y^2 = a^2b^2.$$

The line meets the ellipse where

$$b^2x^2 + a^2(mx + c)^2 - a^2b^2, \text{ that is when}$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0.$$

Since this has only one solution:

$a^4m^2c^2 = (b^2 + a^2m^2)a^2(c^2 - b^2)$

$\quad = a^2b^2c^2 + a^4m^2c^2 - a^2b^4 - a^4b^2m^2$.

Hence $b^2c^2 = b^2(b^2 + a^2m^2)$.

$\therefore\ c^2 = b^2 + a^2m^2$.

$\therefore\ c = \pm\sqrt{a^2m^2 + b^2}$

Corollary: If the line $x \cos\theta + y \sin\theta = p$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

Proof: If $\sin\theta = 0$ this is easily checked.

Suppose that $\sin\theta \neq 0$.
The equation of the line can be written as $y = mx + c$ where
$m = -\dfrac{\cos\theta}{\sin\theta}$ and $c = \dfrac{p}{\sin\theta}$.
Hence $\dfrac{p^2}{\sin^2\theta} = a^2\left(\dfrac{\cos^2\theta}{\sin^2\theta} + b^2\right)$ and so
$p^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

Theorem 4: The equation of the chord of contact of the tangents from $P(x_0, y_0)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

Proof: Let the points of contact be $Q(x_1, y_1)$ and $R(x_2, y_2)$. Then the tangents at Q, R are respectively:

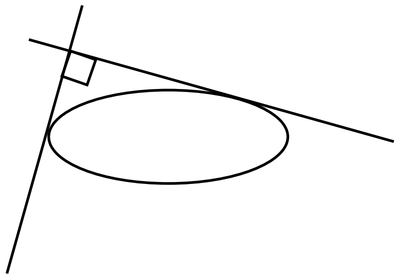
$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \text{ and } \frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$$

Suppose these intersect at $T(x_0, y_0)$. Then

$$\frac{x_0x_1}{a^2} + \frac{y_0y_1}{b^2} = 1 \text{ and } \frac{x_0x_2}{a^2} + \frac{y_0y_2}{b^2} = 1.$$

Hence both Q and R lie on the line $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = 1$ and so this the equation of the chord QR.

Theorem 4: The locus of the points of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the circle $x^2 + y^2 = a^2 + b^2$.
Proof:



A typical line has equation $x \cos \theta + y \sin \theta = p$ and if this is tangent to the ellipse then $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$. A line perpendicular to this has equation

$$- x \sin \theta + y \cos \theta = q, \text{ for some } q.$$

If this is tangent to the ellipse then $q^2 = a^2\sin^2\theta + b^2\cos^2\theta$. Hence, if (x, y) is the point of intersection of these tangents then:

$$x \cos \theta + y \sin \theta = p \text{ and}$$
$$- x \sin \theta + y \cos \theta = q \text{ where}$$
$$p^2 = a^2\cos^2\theta + b^2\sin^2\theta \text{ and}$$
$$q^2 = a^2\sin^2\theta + b^2\cos^2\theta.$$

Squaring and adding the first two equations we get

$$x^2 + y^2 = p^2 + q^2 = a^2 + b^2.$$

§7.3. Parameters

For all θ the point P(a cosθ, b sinθ) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

So (a cos θ, b sin θ) represents a typical point on the ellipse. As θ moves from 0 to 360°, P moves anti-clockwise around the ellipse, starting at (a, 0). The angle θ is called the **eccentric angle** of P.

The **auxiliary circle** for this ellipse is the circle

$$x^2 + y^2 = a^2.$$